Spring 2024, Math 579: Week 0 Problem Set
Due: Thursday, January 24th, 2024
Proof Writing Review

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Fix n, and suppose $A, B \subset \{1, 2, \ldots, n\}$. Prove that $A \cap B = A \cup B$ if and only if $A = B$.

(H2) Suppose $a_0 = 0$, $a_1 = 1$, and for all $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$. Use induction to prove

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for all $n \geq 0$.

(H3) Locate and correct the **error** in the following proof that for any $n \in \mathbb{Z}_{\geq 0}$, there exist $q, r \in \mathbb{Z}_{\geq 0}$ with $0 \leq r \leq 5$ such that $n = 6q + r$.

Proof. Let $P(n)$ denote the following statement:

"There exist $q, r \in \mathbb{Z}_{\geq 0}$ with $0 \leq r \leq 5$ such that $n = 6q + r$."

We proceed by induction on n.

Base cases: suppose $n = 0$, 1, 2, 3, 4, or 5. Choosing $q = 0$ and $r = n$, we see $6q + r = n$. This proves $P(0)$, $P(1)$, $P(2)$, $P(3)$, $P(4)$, and $P(5)$.

Inductive step: supposing $n \geq 6$ and that $P(n-6)$ holds (the *inductive hypothesis*), we wish to prove $P(n)$ holds. Since $P(n-6)$ holds,

$$n - 6 = 6q' + r'$$

for some $q', r' \in \mathbb{Z}$ with $0 \leq r' \leq 5$. Rearranging yields

$$n = 6(q' + 1) + r',$$

and choosing $q = q' + 1$ and $r = r' + 1$ completes the proof that $P(n)$ holds. □

(H4) Determine whether each of the following statements is true or false. Prove your claims (your argument should **not** involve graphs of functions).

(a) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 9x^2 - 6x + 1$ is injective (one-to-one).
(b) The function $f : \mathbb{Z} \to \mathbb{R}$ given by $f(x) = 9x^2 - 6x + 1$ is injective (one-to-one).

(H5) Let $D = \{(a, b) : a, b \in \mathbb{Z}_{\geq 0}\}$, and consider the function $f : D \to \mathbb{Z}_{\geq 0}$ given by

$$f(a, b) = \tfrac{1}{2}(a + b)(a + b + 1) + a.$$

(a) On the xy-plane, mark each point $(a, b) \in D$ with $a + b \leq 6$ with a large dot, and label each dot with the number $f(a, b)$. What do you notice about the labels?
(b) Prove that for each $n \in \mathbb{Z}_{\geq 0}$, there exist $(a, b) \in D$ with $f(a, b) = n$.
Hint: use induction on n.
(c) Prove that if $(a, b), (a', b') \in D$ and $f(a, b) = f(a', b')$, then $a = a'$ and $b = b'$.
Hint: consider the cases $a + b = a' + b'$ and $a + b < a' + b'$ separately.
(d) What do parts (b) and (c) allow us to conclude about f?