Spring 2024, Math 579: Week 6 Problem Set
Due: Thursday, March 7th, 2024
Set Partitions

Discussion problems. The problems below should be worked on in class.

(D1) *A Pascal-like recurrence.* Recall that $S(n,k)$ denotes the number of set partitions of $[n]$ into exactly k blocks (called the *Stirling numbers of the 2nd kind*).

(a) Find $S(5,4)$, $S(4,3)$, and $S(4,4)$ by listing set partitions.

(b) Verify $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ holds for $n = 5$, $k = 4$.

(c) Give a combinatorial proof that, for $n - 1 \geq k \geq 1$, we have

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Hint: count the ways to partition an n-person class *you are in* into k nonempty groups.

(D2) *Bell numbers.* Let $B(n)$ denote the total number of set partitions of $[n]$, that is,

$$B(n) = \sum_{k=1}^{n} S(n,k)$$

(called the n-th *Bell number*).

(a) Find $B(1)$, $B(2)$, $B(3)$, and $B(4)$. Which of these did you find in the prelim problem?

(b) Find $B(5)$ using the identity

$$B(n+1) = 1 + \sum_{k=1}^{n} \binom{n}{k} B(k).$$

(c) Give a combinatorial proof of the identity in part (b).

(d) Give a combinatorial proof that for all $n \geq 1$,

$$B(n+1) - B(n) = \sum_{k=1}^{n} kS(n,k).$$

(D3) *An unfortunate formula for Stirling numbers.* For $n \geq k \geq 1$, let $O_{n,k}$ denote the total number of surjective functions $[n] \to [k]$ (that is, functions in which every element of $[k]$ is the image of some element of $[n]$).

(a) Find **all** functions $[3] \to [2]$. Label each as "surjective" or "not surjective" accordingly.

(b) Find a formula for the total number of functions $[n] \to [k]$.

(c) Show that $O_{5,2} = 2^5 - 2$. Hint: why is it expressed like this?

(d) Argue that $O_{4,3} = 3^4 - \binom{3}{2}2^4 + \binom{3}{1}1^4$.

(e) Using the idea in the previous part, find an expression for $O_{5,3}$.

(f) For $n \geq k$, use the Sieve formula to show that

$$O_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

Hint: first, verify this expression matches what you found in the previous part.

(g) Verify $O_{4,3} = \binom{4}{2} \cdot 3 \cdot 2$. Hint: why is it expressed like this?

(h) Give a combinatorial proof that $S(n,k) = O_{n,k}/k!$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Use the recurrence identity $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$ from class to complete the next 3 rows of the triangle of Stirling numbers of the 2nd kind (the analog of Pascal's triangle where the entry in row n and position k is $S(n,k)$).

Note: you do **not** need to show work for this problem.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 7 & & 6 & & 1 \end{array}$$

(H2) Find a closed formula for $S(n, n-2)$ in terms of n, valid for $n \geq 3$.

(H3) Find a formula for the number of partitions of $[n]$ into blocks of size exactly 2.

(H4) Let $F(n)$ denote the number of set partitions of $[n]$ with no singleton blocks. Prove that $B(n) = F(n) + F(n+1)$.

(H5) Find a recursive formula for $F(n+1)$ in terms of $F(k)$ for $k \leq n$, and give a combinatorial proof of its correctness.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give a combinatorial proof that

$$B(n+m) = \sum_{j=1}^{m} S(m,j) \left(j^n + \sum_{k=1}^{n} \binom{n}{k} j^{n-k} B(k) \right).$$

(C2) Give a combinatorial proof that

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

Hint: Problem (D3) provides a potential path to complete this problem.