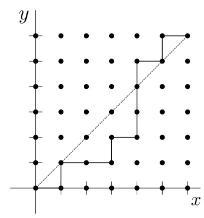
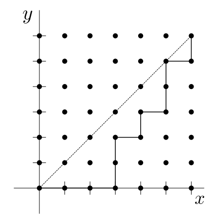
Spring 2024, Math 579: Week 7 Problem Set
Due: Thursday, March 14th, 2024
Catalan Numbers

Discussion problems. The problems below should be worked on in class.

(D1) *A formula for the Catalan numbers.*

(a) A *lattice path* is a path consisting only of unit moves up and right. For example:





Argue the number of lattice paths from $(0,0)$ to (m,n) for $m,n \in \mathbb{Z}_{\geq 0}$ equals $\binom{m+n}{n}$.

(b) A *Dyck path* is a lattice path between $(0,0)$ and (n,n) which does not pass above the line $y=x$. In the above examples, the right-hand lattice path is a Dyck path, but the left-hand lattice path is not. Draw all 5 Dyck paths between $(0,0)$ and $(3,3)$.

(c) Have each group member draw a lattice path $(0,0)$ to $(7,7)$ that is **not** Dyck path. Locate the first up move that occurs above the diagonal, and "flip" the rest of the path after that move (e.g., up moves become right moves and visa versa). Do the same with a lattice path $(0,0)$ to $(10,10)$. What do you notice about the new ending points?

(d) Prove that the number D_n of Dyck paths between $(0,0)$ and (n,n) is given by

$$D_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

(D2) *The Catalan numbers.* Prove each of the following sequences C_n coincides with the Catalan numbers by demonstrating $C_0 = 0$ and giving a combinatorial proof that for each $n \geq 0$,

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

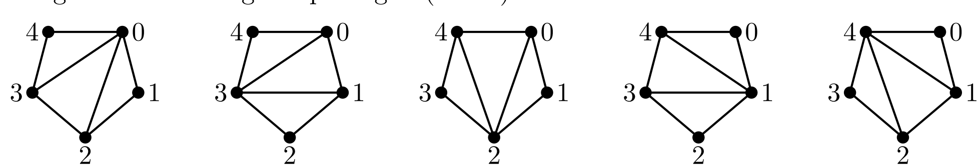
(a) The number C_n of ways of properly parenthesizing the product of $n+1$ values. For example, if $n=3$, then there are 5 ways:

$$((ab)c)d \qquad (ab)(cd) \qquad (a(bc))d \qquad a((bc)d) \qquad a(b(cd)).$$

Hint: you may find it helpful to find all 14 ways to do this when $n=4$.

(b) The number C_n of Dyck paths from $(0,0)$ to (n,n).

(c) The number C_n of *triangulations* of a regular $(n+2)$-gon, that is, the ways of drawing $n-3$ non-crossing diagonals to form $n-2$ triangles. For example, the following are triangluations of a regular pentagon ($n=3$).



Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) We have now seen that each of the following are counted by the Catalan numbers.

(a) Binary trees with n vertices.
(b) Stacks of coins with n coins on the base level.
(c) Dyck paths from $(0,0)$ and (n,n).
(d) Ways of parenthesizing the product of $n+1$ values.
(e) Triangulations of an $(n+2)$-gon.

Pick any 2 of the above, and locate a bijection between them.