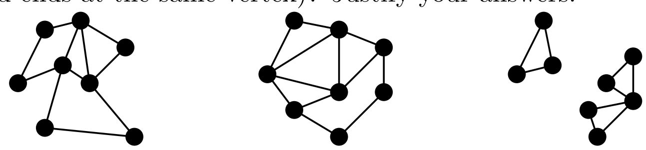
Spring 2024, Math 579: Week 10 Problem Set
Due: Thursday, April 11th, 2024
Introduction to Graphs

Discussion problems. The problems below should be worked on in class.

(D1) *Eulerian walks.* Fix a graph $G = (V, E)$. A walk on G is *Eulerian* if every edge in G is used exactly once.

(a) Which of the following graphs have *closed* Eulerian walk (that is, an Eulerian walk that begins and ends at the same vertex)? Justify your answers.



(b) Have each member of your group draw a graph with at least 8 vertices and 15 edges. Together, determine which of these graphs has a closed Eulerian walk.

(c) Conjecture a criterion for when a simple graph G has a closed Eulerian walk.

(d) Which simple graphs have a (not necessarily closed) Eulerian walk? Try a few first!

(e) Make each of the above graphs *directed* by manually adding a direction to each edge. Now, which graphs have a closed Eulerian walk? Do this several times for each graph.

(f) Conjecture a criterion for when a simple directed graph G has a closed Eulerian walk.

(D2) *Graph automorphisms.* An *automorphism* of a graph G is a bijection $f : V(G) \to V(G)$ such that $(v, w) \in E(G)$ precisely when $(f(v), f(w)) \in E(G)$.

(a) Draw the cycle graph C_4, and label the vertices with $1, 2, 3, 4$ in a clockwise fashion. Write all 8 automorphisms of C_4. As an example, one automorphism f is given by

$$f(1) = 2, \qquad f(2) = 3, \qquad f(3) = 4, \qquad \text{and} \qquad f(4) = 1.$$

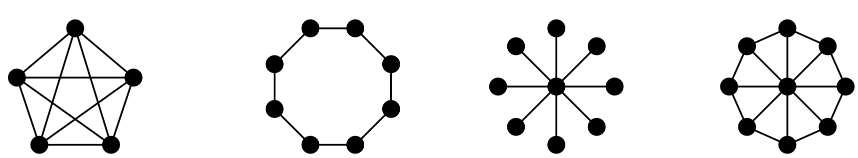
(b) Complete the following proof that for each $n \geq 3$, the n-vertex cycle graph $G = C_n$ has exactly $2n$ automorphisms. Draw an accompanying "graph fragment" picture too.

Proof. Label the vertices of G by $1, 2, \ldots, n$ in a clockwise fashion. For convenience, if we refer to vertex $n + 1$, we mean vertex 1, and similarly for vertex $n + 2$, etc. Suppose $f : V(G) \to V(G)$ is an automorphism of G, and let $v = f(1)$. We must have $f(2) = v + 1$ or ____ since $(1, 2) \in E(G)$ and these are the only vertices connected to v. If $f(2) = v + 1$, then $f(3) =$ ____, $f(4) =$ ____, and so on since __________. Similarly, if $f(2) =$ ____, the remaining values of f must again follow in cyclic order. This means f is determined by choosing the value $f(1)$ (___ possibilities) and then an adjacent value for $f(2)$ (___ possibilities), yielding $2n$ total automorphisms. □

(c) Prove that if $n \geq 2$ and $G = K_n$ is the complete graph with vertex set $[n]$, then **every** bijection $V(G) \to V(G)$ is an automorphism.

(d) Find (with proof!) all automorphisms of the star graph $G = S_n$ with $n \geq 2$ appendages.

(e) Find (with proof!) all automorphisms of the wheel graph $G = W_n$ for $n \geq 3$ spokes.



Complete graph K_5 Cycle graph C_8 Star graph S_8 Wheel graph W_8

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Is there a disconnected simple graph on 7 vertices such that every vertex has degree at least 3?

(H2) Prove that if there is a walk from a vertex v to a vertex w in a simple graph G, then there is a path from v to w.

(H3) Suppose G is a k-regular graph (that is, a simple graph in which every vertex has degree exactly k). Prove that G has a cycle of length at least $k+1$.

(H4) We say two graphs G and G' are *isomorphic* if there exists a bijection $f : V(G) \to V(G')$ such that $(v, w) \in E(G)$ precisely when $(f(v), f(w)) \in E(G')$.

(a) Find 11 graphs, each with 4 vertices, such that no two of the graphs are isomorphic. Be sure to argue that no two of the graphs on your list are isomorphic.

Hint: you may find it helpful to find the degree sequence of each graph in your list. (The *degree sequence* of a graph is an ascending list of the degrees of its vertices; e.g., the degree sequence of S_3 is $(1, 1, 1, 3)$ and the degree sequence of K_4 is $(3, 3, 3, 3)$.)

(b) Argue that every four vertex graph is isomorphic to one on your list from part (a).

Hint: your proof may involve numerous cases, but try to be systematic!

(H5) Find a simple graph G that has no nontrivial automorphisms (that is, where the only automorphism is the identity map).

(H6) Locate two non-isomorphic 3-regular graphs G and G' with the same number of vertices. Conclude that two graphs can have the same degree sequence and still not be isomorphic.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) A *Hamiltonian cycle* is a cycle which visits every vertex exactly once.

If G is a simple graph with n vertices and no Hamiltonian cycles, then what is the maximum number of edges G can have? (Your answer should depend on n.)