Spring 2024, Math 579: Week 12 Problem Set
Due: Thursday, April 25th, 2024
Colorings and Bipartite Graphs

Discussion problems. The problems below should be worked on in class.

(D1) *Chromatic polynomials.* Fix a graph G with $n = |V(G)|$. The *chromatic function* of G is

$$\chi_G(k) = \#\text{ proper colorings of } G \text{ with } k \text{ colors.}$$

(a) Find the chromatic function $\chi_G(k)$ of each of the following graphs G.
  (i) The competely disconnected graph D_n with n vertices and no edges.
  (ii) The graph K_5 with one edge removed.
  (iii) The complete graph K_n.
  (iv) The path graph P_n.

Write each answer as a polynomial in k.

(b) The *complete bipartite graph* is the bipartite graph $K_{\ell,m} = (X, Y)$ with $|X| = \ell$ and $|Y| = m$ such that every vertex of X is adjacent to every vertex of Y.
Find the chromatic function $\chi_G(k)$ where $G = K_{2,2}$ and $G = K_{2,3}$.

(c) Suppose G has n vertices, and let c_i denote the number of ways to properly color G using **exactly** i distinct colors. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \sum_{i=1}^{n} \binom{k}{i} c_i = \binom{k}{1} c_1 + \binom{k}{2} c_2 + \cdots + \binom{k}{n} c_n$$

for all $k \geq 1$. Find c_1, c_2, and c_3 for $G = K_3$, and use this to find $\chi_G(k)$.

(d) Sketch a combinatorial proof of the identity in part (c).

(e) Fix an edge $e = (v, w) \in E(G)$. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \chi_{G \setminus e}(k) - \chi_{G/e}(k).$$

Using **only** this equality and part (a)(i), find the chromatic polynomial of $G = K_3$.

(f) Complete the following combinatorial proof of the equality in part (e).

Proof. Consider the proper colorings of $G \setminus e$ (there are _______ total). For each, either (i) v and w have distinct colors, in which case it is also a proper coloring of G, or (ii) v and w have identical colors, in which case a proper coloring of G/e can be obtained by ______________. As such, $\chi_{G \setminus e}(k) =$ _______ + _______. □

(g) Prove $\chi_G(k)$ is a polynomial using induction on the number of edges of G (this is why we actually call $\chi_G(k)$ the *chromatic polynomial* and not just the *chromatic function*).
Hint: use part (a)(i) as your base case, and part (e) in your inductive step.

(h) Explain briefly why your proof in the previous part guarantees (i) the coefficients of $\chi_G(k)$ are all integers, and (ii) the degree of $\chi_G(k)$ is n.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the chromatic polynomial of each of the following graphs.

(a) The graph G obtained from K_n by removing one edge.

(b) The graph $K_{2,n}$.

(H2) Fix a graph G with n vertices and m edges.

(a) Prove that the leading coefficient of $\chi_G(k)$ (that is, the coefficient of k^n) is 1.

(b) Prove that the coefficient of k^{n-1} in the chromatic polynomial $\chi_G(k)$ equals $-m$.

Hint: each part of this problem can be proven in (at least) 2 distinct ways. One way uses a direct proof with Problem (D1)(c), and another uses induction with Problem (D1)(e).

(H3) Fix a connected graph G with n vertices. Prove that G is a tree if and only if

$$\chi_G(k) = k(k-1)^{n-1}.$$

Hint: use induction for the forward direction and Problem (H2) for the backwards direction.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Solve all parts of Problem (H2) using a **different** method than you already did.