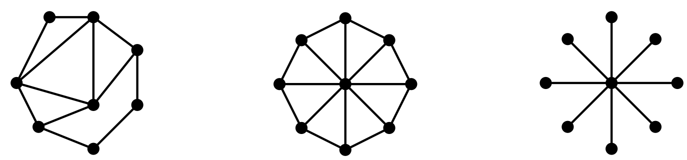
Spring 2024, Math 579: Week 13 Problem Set
Due: Thursday, May 2nd, 2024
Planar Graphs

Discussion problems. The problems below should be worked on in class.

(D1) *Counting faces of planar graphs.* For a planar graph G, let V, E, and F denote the number of vertices, edges, and faces of G, respectively.

- (a) Compute the quantity $V - E + F$ for each of the following graphs.



- (b) Have each group member draw their favorite connected planar graph with at least 8 vertices and 15 edges, and compute $V - E + F$ for their graph.
- (c) Notice this came out the same for each graph. This is known as *Euler's theorem* for planar, connected graphs. We will prove this by induction on E.
    - (i) Base case: prove Euler's theorem when $E = V - 1$. Why is this the base case?
    - (ii) **Carefully and precisely**, write the inductive hypothesis.
    - (iii) What can happen when an edge $e \in E(G)$ is removed?
    - (iv) Finish your proof that Euler's theorem holds for any planar graph G.

(D2) *Duals of planar graphs and a test for planarity.*

- (a) Justify the following claim: if a graph G has m edges and vertices $v_1, \ldots, v_n$, then
$$\deg(v_1) + \cdots + \deg(v_n) = 2m.$$
- (b) Use Euler's Theorem to give a non-pictorial proof that K_5 is not planar.
Hint: how many faces would it have, and how many sides would each need to have?
- (c) Use Euler's Theorem to give a non-pictorial proof that $K_{3,3}$ is not planar.
Hint: is it possible for a face to have 3 boundary edges?
- (d) Fix a **simple** graph G with V vertices and E edges.
    - (a) Prove that if G is planar, then $3F \leq 2E$.
    Hint: what can be said about vertex degrees in G^*?
    - (b) Use the previous part and Euler's theorem to prove if G is planar, then $E \leq 3V - 6$.
    - (c) Is it true that any connected graph satisfying $E \leq 3V - 6$ is planar?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove that if any 2 edges are removed from the graph K_6, the result is not planar. Is the same true if we remove 3 edges?

(H2) Suppose G is a connected planar graph in which every face has at least 4 boundary edge. Prove $E \leq 2V - 4$.

Clarification: the *number of boundary edges* of a face F is the number of edges traversed when walking around the boundary of F. For example, in the cycle graph with 4 vertices, the inside face and outside face each have 4 boundary edges, but in the path graph with 4 vertices, the outside face has 6 boundary edges.