(5) Consider the following improperly written definition.

"A *root* is when a you plug a number into a polynomial and get 0."

Using the above as a guide, write a proper mathematical definition for the term *root*.

Fix a polynomial $f(x)$. We say a number r is a root of $f(x)$ if $f(r)=0$.

(6) Fix arbitrary propositions p, q, and r. Determine whether each of the following equivalences is valid. Prove your claims **without using truth tables**.

(a) $(p \vee q) \wedge r \equiv p \vee (q \wedge r)$

This is not valid. Indeed, if p is T, q is T, and r is F, then $(p \vee q) \wedge r$ is F, but $p \vee (q \wedge r)$ is T.

(b) $(p \to q) \vee r \equiv \neg p \vee (q \vee r)$

This is valid. Indeed we have

$$\begin{aligned} (p \to q) \vee r &\equiv (\neg p \vee q) \vee r && \text{(conditional interpretation)} \\ &\equiv \neg p \vee (q \vee r). && \text{(associativity)} \end{aligned}$$

(7) Fix arbitrary propositions p, q, r. **Without truth tables**, prove the semantic theorem

$$\neg q \to \neg p, p \vee r \vdash q \vee r.$$

Proof: Since $p \vee r$ is T, there are 2 cases.

- If p is T, then by Modus Tollens, since $\neg q \to \neg p$ is T, we know q is T. Addition then yields $q \vee r$ is T.
- If p is F, then by disjunctive syllogism, since $p \vee r$ is T, r must be T. As such, by addition, $q \vee r$ is T.

In each case, we have shown $q \vee r$ is T. ∎

(8) Prove that $\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}, \forall z \in \mathbb{R}, x > yz$.

Proof: fix $x \in \mathbb{N}$. Choose $y = 0$. Then, fixing $z \in \mathbb{R}$, we have

$$x \geq 1 > 0 = y \cdot z,$$

as desired. ∎

(9) Prove or disprove: for all $a, b \in \mathbb{Z}$, if a is odd and b is even, then $a^2 + ab$ is even.

This is false. Indeed if $a = 1$ and $b = 0$, then

$$a^2 + ab = 1 + 0 = 1$$

is odd, even though $a = 2 \cdot 0 + 1$ is odd and $b = 2 \cdot 0$ is even.