(4) Solve the recurrence relation that has $a_0 = 3$, $a_1 = 5$, and $a_n = 10a_{n-1} - 25a_{n-2}$ for each $n \geq 2$.

Characteristic polynomial: $r^2 - 10r + 25$

Solve $r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$

$r_1 = r_2 = 5$

General solution: $a_n = A \cdot 5^n + Bn5^n$

$a_0 = 3 = A$

$a_1 = 5 = 5A + 5B$

$= 15 + 5B$

$B = -2$

$$\boxed{a_n = 3 \cdot 5^n - 2n5^n}$$

(5) Without using any theorems, prove that $2^n + n^2 = O(3^n)$.

Let $M=2$ and $n_0=2$. Fix $n \geq n_0$. We have

$|2^n + n^2| = 2^n + n^2 \leq 2^n + 2^n \leq 3^n + 3^n = 2 \cdot 3^n = 2|3^n|$,

as desired. ☐

(6) Fix $x \in \mathbb{R}$. Prove that if $x - \lfloor x \rfloor < \frac{2}{3}$, then $\lfloor x + \frac{1}{3} \rfloor = \lfloor x \rfloor$.

Fix $x \in \mathbb{R}$.

Suppose $x - \lfloor x \rfloor < \frac{2}{3}$. This means

$\lfloor x \rfloor \leq x \leq x + \frac{1}{3} < \lfloor x \rfloor + 1$,

where the first inequality follows from a theorem in class (5.17).

As such, by the definition of floors,

$\lfloor x + \frac{1}{3} \rfloor = \lfloor x \rfloor$. ☐

(7) Prove that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

We proceed by induction on n.

Base Case: $n=1$. We see $\sum_{i=0}^{1} 2^i = 2^0 + 2^1 = 3$ and $2^{n+1} - 1 = 3$.

Inductive Step: fix $n \in \mathbb{N}$ and suppose $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

We wish to show $\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1$.

We see

$$\sum_{i=0}^{n+1} 2^i = 2^{n+1} + \sum_{i=0}^{n} 2^i = 2^{n+1} + 2^{n+1} - 1$$

$$= 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1,$$

as desired. ☐

(8) Prove or disprove: $\exists! x \in \mathbb{R}, \exists! y \in \mathbb{R}, y^2 = x$.

This statement is true.

We first prove existence of $x \in \mathbb{R}$ such that $\exists! y \in \mathbb{R}, y^2 = x$.

Choosing $x=0$, we see the unique solution to $y^2 = 0$ is $y=0$.

Next, we prove uniqueness of $x \in \mathbb{R}$. Suppose $x \neq 0$. We consider cases.

- If $x > 0$, then the equation $y^2 = x$ has 2 solutions, so y is not unique.
- If $x < 0$, then the equation $y^2 = x$ has no solutions, so y does not exist.

As such, $x=0$ is the ~~only~~ unique real value ~~that~~ such that $\exists! y \in \mathbb{R}, y^2 = x$ is true. ☐