Math 245 - Spring 2026
Instructor: Christopher O'Neill
Friday, March 13th, 2026
Midterm Exam 2

Name: __________________________________ **Red ID:** ________________________

Please write legibly, with plenty of white space. Please **print** your name and Red ID in the designated spaces above. Be sure to read problem directions carefully. Please fit your answers into the designated areas; material outside the designated areas (such as on this cover page) will not be graded. Each problem is worth 10-20 points, for a maximum score of 100. The use of notes, books, calculators, or other materials on this exam is strictly prohibited, except you may bring one 3"x5" card (both sides) with your handwritten notes. This exam will begin at 10:00 and will end at 10:50; pace yourself accordingly.

Logistical instructions:

(1) Stow all bags/backpacks/purses at the front of the room. All contraband, except phones, must be stowed in your bag. All smartwatches and phones must be silent, non-vibrating, and either in your pocket or stowed in your bag.
(2) Please remain quiet to ensure a good test environment for others.
(3) Please keep your exam on your desk; do not lift it up for a better look.
(4) If you have a question or need to use the restroom, please come to the front. Bring your exam. I cannot come to you unless you are sitting by an aisle.
(5) If you are done and want to submit your exam and leave, please wait until one of the three designated exit times below. Please do **NOT** leave at any other time. If you are sure you are done, just sit and wait until the next exit time, with this cover sheet visible.

Good luck!

Designated exit times: 10:20 10:40 10:50

(1) Fill in each of the following blanks. No justification is required for this problem.

- Fix $x \in \mathbb{R}$ and a set T of integers. If $x \leq y$ for all $y \in T$, then x is __________________________ of T. If, additionally, _______________________, then x is _____________________________ of T.
- A sequence a_n for which ____ terms are defined in terms of previous terms is called a ____________________________________.

  If $a_n = 2a_{n-1} + 3a_{n-3}$ for all $n \geq 7$, then the order is ____________.
- To prove "$\forall n \in \mathbb{Z}$ with $n \geq 4$, $2^n > 6$" using _______________ and _______________ induction, we prove (i) $2^n > 6$ for $n =$ _____, and (ii) $\forall n \in \mathbb{Z}$ with $n \geq$ _____, we have $2^{n-1} > 6 \longrightarrow 2^n > 6$.
- We have $5n^2 + 8n + 17$_____$O(3n^3)$, $\quad 5n^2 + 2^n + 17^n$_____$O(5^n)$, $\quad$ and $2^n + 3^n + 4^n$_____$\Theta(n^n)$.
- The negation of "$\exists x \in \mathbb{R}, !y \in \mathbb{R}, x < y$" is ____.

(2) Consider the following theorem.

> For every $n \in \mathbb{Z}$, the following are equivalent:
> (a) n is even; (b) $n + 1$ is odd; (c) $n - 1$ is odd; and (d) $n + 2$ is even.

For each of the following, write "valid" if proving the specified list of statements would constitute a valid proof, and write "invalid" next to each list that would not. No justification is required.

_______________________ (a) $\vdash$ (c), (b) $\vdash$ (d), (c) $\vdash$ (b), (d) $\vdash$ (a)

_______________________ (a) $\vdash$ (b), (b) $\vdash$ (c), (c) $\vdash$ (a), (a) $\vdash$ (d), (d) $\vdash$ (b)

_______________________ (a) $\vdash$ (b), (b) $\vdash$ (a), (c) $\vdash$ (d), (d) $\vdash$ (c), (d) $\vdash$ (b)

_______________________ (a) $\vdash$ (b), (b) $\vdash$ (d), (d) $\vdash$ (c), (a) $\vdash$ (c)

(3) Fill in the blanks in the following proof that for all $n \in \mathbb{N}$, $5^n > 2^{n+1}$.

Proof: Let S denote the set of all $n \in \mathbb{N}$ for which ____________________. If S is empty, we are done.

As such, by way of contradiction, suppose _______________________. Since $1 \leq n$ for every $n \in S$,

by ____ we know S has a _______________ element $m \in S$.

Since $5^1 = 5 > 4 = 2^{1+1}$, we know $m \neq$ ________, so since $m \in \mathbb{N}$, we have ________ $\in \mathbb{N}$. Moreover,

$$5^{m-1} = \tfrac{1}{5}5^m \leq \tfrac{1}{5}2^{m+1} \leq \tfrac{1}{2}2^{m+1} = 2^m,$$

so we conclude ________ $\in S$. This contradicts ____.

(4) Solve the recurrence relation that has $a_0 = 3$, $a_1 = 5$, and $a_n = 10a_{n-1} - 25a_{n-2}$ for each $n \geq 2$.

(5) Without using any theorems, prove that $2^n + n^2 = O(3^n)$.

(6) Fix $x \in \mathbb{R}$. Prove that if $x - \lfloor x \rfloor < \frac{2}{3}$, then $\lfloor x + \frac{1}{3} \rfloor = \lfloor x \rfloor$.

(7) Prove that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

(8) Prove or disprove: $\exists! x \in \mathbb{R}$, $\exists! y \in \mathbb{R}$, $y^2 = x$.