Discussion problems. The problems below should be worked on in class.

(D1) *Manipulating power series.* In what follows, denote

$$A(z) = \sum_{n \geq 0} a_n z^n \qquad \text{and} \qquad B(z) = \sum_{n \geq 0} b_n z^n.$$

    (a) For each of the following, find an expression for $A(z)$ that has no sigma-sums. You may find it helpful to reference the second centered equation in Problem (D2).

        (i) $a_n = 4n + 3$
        (ii) $a_n = 2n + 1$ if n is even, $a_n = 3n - 1$ if n is odd.

    (b) Find a formula for a_n in terms of n given the following $A(z)$.

        (i) $A(z) = \dfrac{3z + 17}{1 - z^2}.$
        (ii) $A(z) = \dfrac{3z + 17}{(1 - z)^2}.$

    (c) For each of the following $A(z)$, find a formula for the coefficients of $B(z) = 1/A(z)$. Then, prove via direct multiplication (using coefficients) that $A(z)B(z) = 1$.

        (i) $A(z) = 1 - 3z + 2z^2$
            Hint: find the first few coefficients of $B(z)$ by hand, conjecture a formula for b_n.
        (ii) $A(z) = \sum_{n \geq 0} (-1)^n z^n$
            Hint: start by writing $A(z)$ as a rational function.

(D2) *Power series of polynomial functions.* Given a power series $A(z) = \sum_{n \geq 0} a_n z^n$, define its "formal derivative" as

$$A'(z) = \frac{d}{dz} A(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n.$$

    (a) Applying formal differentation to the power series $1 + z + z^2 + \cdots$ and "quotient rule" to $1/(1 - z)$ yields

$$1 + 2z + 3z^2 + \cdots = \sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1 - z)^2}.$$

    Verify the equality using power series multiplication.

    (b) Manipulate the above expression to write $\sum_{n=0}^{\infty} n z^n$ as a rational expression in z.

    (c) Use "formal differentiation" to express $\sum_{n=0}^{\infty} n^2 z^n$ as a rational expression in z.

    (d) Use "formal differentiation" to express $\sum_{n=0}^{\infty} n^3 z^n$ as a rational expression in z.

    (e) Give a proof by induction that for each $d \geq 1$,

$$\sum_{n=0}^{\infty} n^d z^n = \frac{Q_d(z)}{(1 - z)^{d+1}}$$

    for some polynomial $Q_d(z)$.
    Hint: start by identifying $Q_1(z)$, $Q_2(z)$, and $Q_3(z)$ from parts (b), (c), and (d).

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find a rational expression for $A(z) = \sum_{\geq 0} a_n z^n$ in terms of z for each of the following sequences a_n. Your answer should **not** contain any "$\cdots$" or sigma-sums.

Hint: you can substantially shorten the algebra needed to obtain your answer with clever use of power series we have already encountered.

    (a) $a_n = n^3 + 7n^2 + 3n - 5$

    (b) $a_n = n2^n - n^2 3^n$

(H2) Find a formula for a_n for each of the following power series $A(z) = \sum_{n \geq 0} a_n z^n$.

Note: your formula for a_n should be in terms of n and have no sigma-sums in it!

    (a) $A(z) = \dfrac{1}{2 + z}$

    (b) $A(z) = \dfrac{5 + 2z + z^2}{1 - z^3}$

    (c) $A(z) = \dfrac{z^2 + 1}{(1 - z)^3}$

        Hint: start by writing this series as a combination of those from Problem (D2).

(H3) Let $A(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$. Prove that $A(z)$ has a multiplicative inverse if and only if $a_0 \neq 0$. Hint: if $A(z)B(z) = 1$ for some $B(z) = b_0 + b_1 z + b_2 z^2 + \cdots$, show each b_n is determined by $a_0, a_1, \ldots, a_n$.

(H4) Define $c(n) = 1$ if there exists a way to write n as a sum of the values 3 and 5, and $c(n) = 0$ otherwise. For instance, $c(13) = 1$ since $13 = 3 + 5 + 5$, and $c(15) = 1$ since $15 = 5 + 5 + 5$ (as well as $15 = 3 + 3 + 3 + 3 + 3$), but $c(7) = 0$ since there is no way to add 3's and 5's together to obtain 7. Prove that

$$\sum_{n=0}^{\infty} c(n) z^n = \frac{1 - z^{15}}{(1 - z^3)(1 - z^5)}.$$

Note: $c(0) = 1$ since $0 = 0 \cdot 3 + 0 \cdot 5$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Conjecture and prove an analogous result to Problem (H4) where positive integers n_1 and n_2 replace 3 and 5.