

**Spring 2026, Math 590: Week 4 Problem Set**  
**Due: Wednesday, February 25th, 2026**  
**Quasipolynomials and Rational Generating Functions**

**Discussion problems.** The problems below should be worked on in class.

(D1) *Eventual polynomials.* Throughout this problem, suppose  $d \geq 0$ , and

$$A(z) = \sum_{n \geq 0} a_n z^n = \frac{Q(z)}{(1-z)^{d+1}}.$$

We proved in class  $a_n$  is a polynomial in  $n$  of degree  $d$  iff  $\deg Q(z) < d+1$  and  $Q(1) \neq 0$ .

- (a) Find a formula for  $a_n$  in terms of  $n$  if  $d = 2$  and  $Q(z) = 5 - 6z + z^2$ . Is the degree of  $a_n$  as expected by the theorem from class?
- (b) Find  $Q(z)$  when  $d = 0$  and  $a_n = n$ . Why does this not violate the theorem from class?
- (c) Find a formula for  $a_n$  in terms of  $n$  if  $d = 2$  and  $Q(z) = 1 + z + z^2 + z^3 + z^4$ .  
Hint: begin by using polynomial long division.
- (d) Based on your answer to the previous part, conjecture a mathematical definition of “eventually polynomial” so that the following theorem holds.

**Theorem.** *If  $d \geq 0$ , then  $Q(z)$  is a polynomial if and only if  $a_n$  is eventually polynomial of degree at most  $d$ .*

- (e) Find  $d$  and  $Q(z)$  when  $a_2 = 17$  and  $a_n = n^2$  for  $n \neq 2$ .
- (f) Prove the theorem in part (d).

Hint: use polynomial long division and the theorem at the start of this problem.

(D2) *Quasipolynomials.* In this problem, suppose  $Q(z)$  is a polynomial,  $d \geq 0$ ,  $p \geq 1$ , and

$$A(z) = \sum_{n \geq 0} a_n z^n = \frac{R(z)}{(1-z^p)^{d+1}}.$$

We saw in class that  $a_n$  is a quasipolynomial in  $n$  of degree at most  $d$  and period dividing  $p$  if and only if  $\deg R(z) < p(d+1)$ .

- (a) Find  $d$ ,  $p$ , and  $R(z)$  when

$$a_n = \begin{cases} \frac{1}{4}n^2 + n + 1 & \text{if } n \equiv 0 \pmod{2}; \\ \frac{1}{4}n^2 + n + \frac{3}{4} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

- (b) Find a formula for  $a_n$  if  $d = 3$ ,  $p = 2$ , and  $R(z) = z + 2z^2 + 3z^3 + 4z^4 + 5z^5$ .
- (c) Prove the theorem at the beginning of this problem.

Hint: start by proving the forward direction, and use the theorem from the beginning of Problem (D1) in your proof. Then, for the backward direction, use linear algebra (and don't hesitate to ask questions!).

- (d) Develop (and prove) an analog of the theorem in Problem (D1)(c) for when  $a_n$  is “eventually quasipolynomial” in  $n$ .

**Homework problems.** You must submit *all* homework problems in order to receive full credit.

- (H1) This problem concerns the theorem you obtained in Problem (D1)(d), with the following definition: we say  $a_n$  is *eventually polynomial* if there exists a polynomial  $f(n)$  and an integer  $N$  such that  $a_n = f(n)$  for all  $n \geq N$ .

Given  $Q(z)$  and  $d$ , describe a method of determining

- (a) the degree of  $f(n)$ ,
- (b) the leading coefficient of  $f(n)$ , and
- (c) the smallest integer  $N$  such that  $a_n = f(n)$  for all  $n \geq N$ .

Be sure to prove your claim in each part.

- (H2) Resume notation from Problem (H1). Prove or disprove: if  $Q(z)$  has integer coefficients, then  $f(n)$  has integer coefficients. What about the converse?
- (H3) Prove that  $a_n$  is a quasipolynomial of degree  $d$ , period dividing  $p$ , and constant leading coefficient if and only if

$$\sum_{n \geq 0} a_n z^n = \frac{Q(z)}{(1-z)(1-z^p)^d}$$

for some polynomial  $Q(z)$  with  $\deg Q(z) < pd + 1$  and  $Q(1) \neq 0$ .

- (H4) We saw in class that if

$$\sum_{n \geq 0} a_n z^n = \frac{1}{(1-z^3)(1-z^4)(1-z^5)}$$

then  $a_n$  equals the number of ways to write  $n$  as a sum of 3's, 4's, and 5's (called *restricted partitions* of  $n$ ). Prove  $a_n$  is a quasipolynomial function of  $n$  with constant leading coefficient. Find its degree, period, and leading coefficient.

Hint: use Problem (H3) and the fact that  $1 - z^4$  has (complex) roots  $1, -1, i,$  and  $-i$ . In particular, do **not** find a formula for  $a_n$  (you will see why once you find the period).

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

- (C1) Fix  $m_1, \dots, m_k \in \mathbb{Z}_{\geq 1}$  and distinct nonzero  $r_1, \dots, r_k \in \mathbb{C}$ . Prove that if  $Q(z)$  is any polynomial with  $\deg Q(z) < m_1 + \dots + m_k$ , we can find constants  $c_{j,\ell} \in \mathbb{C}$  such that

$$\frac{Q(z)}{(1-r_1 z)^{m_1} \dots (1-r_k z)^{m_k}} = \sum_{j=1}^k \left( \frac{c_{j,1}}{1-r_j z} + \frac{c_{j,2}}{(1-r_j z)^2} + \dots + \frac{c_{j,m_j}}{(1-r_j z)^{m_j}} \right).$$

Note: though notationally dense, this is just *partial fraction decomposition* of the left hand side, written out formally.