Spring 2026, Math 590: Week 8 Problem Set
Due: Wednesday, April 8th, 2026
Ehrhart Functions and Ehrhart Series

Discussion problems. The problems below should be worked on in class.

(D1) *Computing Ehrhart series.* Recall from lecture that if $P \subseteq \mathbb{R}^d$ is a polytope, then

$$\mathrm{Ehr}_P(z) = \sum_{n \geq 0} L_P(n) z^n$$

is the *Ehrhart series* of P.

(a) Recall the *cone over* P, denoted cone(P), is the cone $C = \mathrm{span}_{\geq 0}(P \times \{1\}) \subseteq \mathbb{R}^{d+1}$. Identify which polytope the cone in Problem (P1) is the cone over.

(b) Find the Ehrhart function of the polytope P in part (a). Compre this to Problem (P3).

(c) More generally, suppose $P \subseteq \mathbb{R}^d_{\geq 0}$ is a polytope, and let $C = \mathrm{cone}(P) \subseteq \mathbb{R}^{d+1}_{\geq 0}$. Considering the series $\sigma_C(z_1, \ldots, z_d, z_{d+1})$, give a **geometric explanation** as to why

$$\sigma_C(1, \ldots, 1, z) = \mathrm{Ehr}_P(z),$$

and in particular why the coefficients of z on the left are finite.

(d) Fix integers $a < b$, and let $P = [a, b] \subseteq \mathbb{R}$, that is, the closed interval from a to b (this is a 1-dimensional polytope with vertices a and b). **Using part (c)** and a fundamental parallelopiped, find the Ehrhart series of P in terms of a and b.

(e) Let $P = \mathrm{conv}\{(0, 2), (1, 0), (1, 1)\}$. Draw P, $2P$, and $3P$ on **different** sets of axes. Also draw the cross sections $x_3 = 1$, $x_3 = 2$, and $x_3 = 3$, respectively, of the fundamental parallelopiped of $C = \mathrm{cone}(P)$ (use a different color).

Note: each cross section should be a contained within the corresponding dilation.

(f) Find $\mathrm{Ehr}_P(z)$ for the polytope P in the previous part **using part (c)**.

(g) Consider the rational polygon $P = \mathrm{conv}\{(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2})\}$. **Using part (c)**, show

$$\mathrm{Ehr}_P(z) = \frac{1 + z}{(1 - z^2)^3}.$$

Use this to find a formula for $L_P(t)$ (remember, this will be a *quasipolynomial*).

(h) Find the Ehrhart series of the pentagon

$$P = \mathrm{conv}\{(-1, -1), (1, -1), (-1, 0), (1, 0), (0, 1)\}$$

by triangulating and then using part (c). Use this to find a formula for the Ehrhart polynomial $L_P(t)$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the Ehrhart series of the 2-dimensional permutohedron

$$P_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

Hint: locate a projection of P_3 into $\mathbb{R}^2$ that preserves the Ehrhart function, then triangulate.

(H2) Fix $h \in \mathbb{Z}_{\geq 1}$, and consider

$$T = \{(0,0,0),(1,0,0),(0,1,0),(1,1,h)\},$$

known as *Reeve's tetrahedron*.

(a) Find $\text{Ehr}_T(z)$, the Ehrhart series of Reeve's tetrahedron (your answer will have h's).

(b) Use part (a) to find a formula for $L_T(t)$, the Ehrhart function of T.

(H3) Fix two polytopes $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$, and define

$$P \times Q = \{(p_1,\ldots,p_n,q_1,\ldots,q_m) : (p_1,\ldots,p_n) \in P, (q_1,\ldots,q_m) \in Q\} \subseteq \mathbb{R}^{n+m}.$$

(a) Prove $P \times Q$ is a polytope by finding either its V-description or its H-description.

(b) Express $L_{P\times Q}(t)$ in terms of $L_P(t)$ and $L_Q(t)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix $a, b \in \mathbb{Z}_{\geq 1}$ with $\gcd(a,b) = 1$, and consider the rhombus

$$P = \{(x,y) : a|x| + b|y| \leq ab\}.$$

Find $L_P(t)$ in terms of a, b, and t.