

Spring 2026, Math 590: Week 9 Problem Set
Catch-up Week

Discussion problems. The problems below should be worked on in class.

(D1) *Practice with power series manipulation.* Consider the series

$$\sum_{n \geq 0} (n^3 + 2n^2 + 2n + 1)z^n = \frac{Q(z)}{(1-z)^{d+1}} \quad \text{and} \quad \sum_{n \geq 0} f(n)z^n = \frac{1 + 3z + 4z^2}{(1-z)^3}$$

- (a) Find $Q(z)$ and d .
- (b) Find a formula for $f(n)$.
- (c) Find a formula for $f(n)$ if the denominator on the right hand side were instead $(1-z^2)^3$.

(D2) *Practice with Ehrhart series.* Consider the polytope

$$P = \text{conv}\{(1, 0, 0), (0, 1, 0), (-1, -1, 0), (0, 0, 1), (0, 0, -1)\}.$$

- (a) Divide P into 2 simplices. Use cones over those simplices to obtain $\text{Ehr}_P(z)$.
Hint: what is the issue with points with negative entries? How can we handle this?
- (b) Find the leading coefficient of $L_P(t)$ (**without** using Ehrhart's theorem).
Hint: this can be done without finding $L_P(t)$ in its entirety!

(D3) *Practice with V - and H -descriptions.* Refer to the polytope P from Problem (D1).

- (a) Determine, using the definition of convex hull, which points in the convex hull above can be omitted. Conclude you have obtained the irredundant V -description of P .
- (b) Find the irredundant H -descriptions of P . Identify a facet for each inequality in your H -description (thus ensuring irredundancy).
- (c) Verify from first principles that your V - and H -descriptions are indeed equivalent.

(D4) *Ehrhart series of the cross polytope.* Given a polytope $P = \text{conv}\{v_1, \dots, v_k\} \subseteq \mathbb{R}^d$ with the origin in its interior, define

$$\begin{aligned} \text{Pyr}(P) &= \text{conv}\{(v_1, 0), (v_2, 0), \dots, (v_k, 0), e_{d+1}\} \subseteq \mathbb{R}^{d+1} \quad \text{and} \\ \text{Bipyr}(P) &= \text{conv}\{(v_1, 0), (v_2, 0), \dots, (v_k, 0), e_{d+1}, -e_{d+1}\} \subseteq \mathbb{R}^{d+1}, \end{aligned}$$

the *pyramid* and *bipyramid* over P , respectively (each lives in one dimension more than P).

- (a) Let $P = [-1, 1] \subseteq \mathbb{R}$. Draw P , $\text{Pyr}(P)$ and $\text{Bipyr}(P)$.
- (b) Let $P = \text{conv}\{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Draw P , $\text{Pyr}(P)$ and $\text{Bipyr}(P)$.
- (c) Find a general formula for $L_{\text{Pyr}(P)}(3)$ in terms of $L_P(1)$, $L_P(2)$, and $L_P(3)$.
- (d) Let $A(z) = \sum_{n \geq 0} a_n z^n$ and $B(z) = \sum_{n \geq 0} b_n z^n$, and suppose each $b_n = a_0 + \dots + a_n$. Show that $(1-z)B(z) = A(z)$.
- (e) Find a formula for $\text{Ehr}_{\text{Pyr}(P)}(z)$ in terms of $\text{Ehr}_P(z)$ involving no sigma-sums.
- (f) Find a sigma-sum-free equation relating the Ehrhart series of P , $\text{Pyr}(P)$, and $\text{Bipyr}(P)$. Use this to obtain a formula for $\text{Ehr}_{\text{Bipyr}(P)}(z)$ in terms of $\text{Ehr}_P(z)$.
- (g) Find a formula for the Ehrhart series of the d -dimensional cross polytope

$$P_d = \text{conv}\{e_1, -e_1, e_2, -e_2, \dots, e_d, -e_d\} \subseteq \mathbb{R}^d.$$

- (h) Why do we require that P have the origin in its interior?