Winter 2017, Math 148: Week 2 Problem Set
Due: Wednesday, January 25th, 2017
Groups, Rings, and Fields

Discussion problems. The problems below should be completed in class.

(D1) *Rings of functions.* Determine which of the following sets $(R, +, \cdot)$ forms a ring under the given addition and multiplication. For each ring R, determine whether R is a field.

(a) The set $R = \{\ldots, -2, 0, 2, \ldots\} \subset \mathbb{Z}$ of even integers, under the usual addition and multiplication of integers.

(b) The set $R = \{r_n x^n + \cdots + r_1 x + r_0 : r_i \in \mathbb{R}\}$ of polynomials in a variable x with real coefficients, under the usual addition and multiplication.

(c) The subset of R from part (b) consisting of those polynomials with degree at most d.

(d) The set $R = \{f(x)/g(x) : f(x), g(x) \text{ polynomials}, g \neq 0\}$ of rational functions under the usual addition and multiplication of fractions. (Here, cancelling common factors in the numerator and denominator yields the same element of R.)

(e) The subset of R in part (d) whose denominator takes only positive values over $\mathbb{R}$.

(f) The set $R = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of continuous real-valued functions on $\mathbb{R}$, under the usual addition and multiplication of functions.

(D2) *Zero divisors.* A nonzero ring element $r \in R$ is called a *zero divisor* if there exists a nonzero element $r' \in R$ with $r \cdot r' = 0$.

(a) For which n does the ring $\mathbb{Z}_n$ have zero divisors?

(b) Which rings from problem (D1) have zero divisors?

(c) Give an example of a ring with no zero divisors that is not a field.

(D3) *Gaussian integers.* Consider the ring $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ under usual addition and multiplication of complex numbers (i.e. using $i^2 = -1$).

(a) Verify that $\mathbb{Z}[i]$ is a ring. Is it a field? Which elements are invertible?

(b) Verify that the set U of units of $\mathbb{Z}[i]$ forms a group under multiplication. Which (more familiar) group is this?

(c) Does $\mathbb{Z}[i]$ have any zero divisors?

(D4) *Cartesian products.* Recall that the Cartesian product of two rings R_1 and R_2 is the set

$$R_1 \times R_2 = \{(a, b) : a \in R_1, b \in R_2\}$$

with addition $(a, b) + (a', b') = (a + a', b + b')$ and multiplication $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$.

(a) How many elements does $\mathbb{Z}_n \times \mathbb{Z}_m$ have?

(b) What is the characteristic of $\mathbb{Z}_n \times \mathbb{Z}_m$? What about $R_1 \times R_2$ for rings R_1, R_2?

(c) The *order* of an element $a \in R$ is the smallest integer k such that adding a to itself k times yields 0 (or, written more compactly, $ka = 0$).

(i) Which elements of $\mathbb{Z}_5 \times \mathbb{Z}_3$ have highest order? What about $\mathbb{Z}_6 \times \mathbb{Z}_4$? $\mathbb{Z}_n \times \mathbb{Z}_m$?

(ii) Express the order of $(a, b) \in R_1 \times R_2$ in terms of the orders of $a \in R_1$ and $b \in R_2$.

(iii) Is there a relationship between the order of $a \in R$ and the characteristic of R?

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Find all Abelian groups with exactly 4 elements. Argue that there are no more.

(R2) Fix a field F. Prove that the set $F \setminus \{0\}$ forms a group under multiplication. Is the same true if F is any ring?

(R3) Show that the group $(\mathbb{Z}_{13} \setminus \{0\}, \cdot)$ is cyclic.

(R4) Fix a ring R. Prove that if $r \in R$ is a zero-divisor, then it is not invertible.

Selection problems. You are required to submit all parts of *one* selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Fix two rings R_1 and R_2.

- (a) Characterize the units of $R_1 \times R_2$ in terms of the units of R_1 and R_2.
- (b) Characterize the zero divisors of $R_1 \times R_2$ in terms of the zero divisors of R_1 and R_2.

(S2) Under what conditions on m and n is the group $(\mathbb{Z}_n \times \mathbb{Z}_m, +)$ cyclic? Prove your assertion.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give an example of a ring in which the set of non-unit elements is closed under both addition and multiplication.