Winter 2017, Math 148: Week 7 Problem Set
Due: Wednesday, March 1st, 2017
Block Designs

Discussion problems. The problems below should be completed in class.

(D1) *Building block designs from block designs.* Consider the following 12 subsets of $\{1, \ldots, 9\}$.

$$\begin{array}{cccc} \{1, 2, 3\} & \{1, 4, 7\} & \{1, 5, 9\} & \{1, 6, 8\} \\ \{4, 5, 6\} & \{2, 5, 8\} & \{2, 6, 7\} & \{2, 4, 9\} \\ \{7, 8, 9\} & \{3, 6, 9\} & \{3, 4, 8\} & \{3, 5, 7\} \end{array}$$

- (a) For which t is the above a t-design? What are the values of v, k, r and b in each case?
- (b) Replace each set in the above example with its complement in $\{1, \ldots, 9\}$. For which t does this form a t-design? These are called *complementary designs*. What are the values of v, k, r and b in each case?
- (c) Based on your observation in part (c), state a conjecture for arbitrary 2-designs. Don't worry about proving your conjecture; you will do this on your homework!
- (d) Find an explicit formula for v, k, r and b in part (d) case (your formulas will depend on v, k, r and b in the original example).

(D2) *Designs from finite fields.* The goal of this problem is to prove the following theorem.

Theorem. *If $q = p^r$ for p prime and $r \geq 1$, there is a 2-design with $v = q^2$, $k = q$, $r = 1$.*

- (a) Let's consider the case $q = 3$. Draw the 2-dimensional vector space $V = \mathbb{Z}_3^2$ over $\mathbb{Z}_3$.
- (b) Find all lines (not necessarily containing the origin) in $\mathbb{Z}_3^2$, written as sets of points. You may find it easier to shorten points from $(2, 1)$ to simply "21" in your list.
- (c) Does this collection of sets constitute a 2-design? What are v, k, r, and b?
- (d) We now generalize the above construction from $\mathbb{Z}_3$ to any finite field. Suppose $\mathbb{F}_q$ is a field with q elements, let $V = \mathbb{F}_q^2$ denote a 2-dimensional vector space over $\mathbb{F}_q$, and let $B_1, \ldots, B_b \subset V$ denote the set of lines in V. For these to form a 2-design $(q^2, q, 1)$,
  - (i) each block must have the same size q,
  - (ii) every element must lie in the same number of blocks, and
  - (iii) any pair of elements must occur together in exactly 1 block.

  Restate each of the above requirements for $B_1, \ldots, B_b$ in terms of geometry.
- (e) Any line $L \subset V$ can be written as follows for some $a, b, c \in \mathbb{F}_q$:

$$L = \{(x, y) \in V : ax + by = c\}$$

  Prove that any two lines with more than one point in common must be equal as sets. Hint: given $(x_1, y_1), (x_2, y_2) \in L$ distinct, express a, b and c terms of x_1, y_1, x_2 and y_2.
- (f) Prove that any line $L \subset V$ contains exactly q points.
- (g) Which requirement listed in part (d) follows from the other two? With this in mind, find the number of lines containing each point of V (in terms of q).

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) For each triple (v, k, r) below, construct a block design (1-design) with parameters (v, k, r), or prove that no such design exists.

(a) $(v, k, r) = (7, 6, 6)$

(b) $(v, k, r) = (6, 3, 1)$

(c) $(v, k, r) = (5, 2, 1)$

(d) $(v, k, r) = (9, 6, 4)$

(R2) Suppose that there exists a 5-design with parameters $(v, k, r) = (12, 6, 1)$. Such a design is also an s-design for any $s \leq 5$; find the corresponding parameters for each s. How many blocks must this design have?

(R3) Find a 5-design with parameters $(v, k, r) = (6, 5, 1)$, or argue that no such design exists.

(R4) Is it possible there exists a 3-design with parameters $(v, k, r) = (15, 6, 2)$? What about a 4-design with parameters $(v, k, r) = (11, 5, 1)$?

(R5) Find a 2-design with parameters $(v, k, r) = (16, 4, 1)$ using the theorem in problem (D2). Hint: what field should you use, and what do the elements of that field look like?

Selection problems. You are required to submit all parts of *one* selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Fix a 2-design $B_1, \ldots, B_b \subset \{1, \ldots, v\}$ with parameters (v, k, r), and let $B_i^c = \{1, \ldots, v\} \setminus B_i$ for each $i \leq b$. Prove that the blocks $B_1^c, \ldots, B_b^c$ also form a 2-design. Is the complement of a 3-design always a 3-design?

(S2) Fix a 2-design of the form $(v, 3, 1)$. Prove that v must have the form $6n + 1$ or $6n + 3$ for some $n \geq 0$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix a finite field $\mathbb{F}_q$, and let $V = \mathbb{F}_q^3$ denote a 3-dimensional vector space over $\mathbb{F}_q$. A *hyperplane* is a subset $H \subset V$ given by

$$H = \{(x, y, z) \in V : ax + by + cz = d\}$$

for some $a, b, c, d \in \mathbb{F}_q$ (in particular, note that a hyperplane need not contain the origin).

(a) Find the number of hyperplanes in V (as a function of q).

(b) Determine for which t the collection of all hyperplanes in V forms a t-design.