Winter 2018, Math 148: Week 2 Problem Set
Due: Wednesday, January 24th, 2018
Groups, Rings, and Fields

Discussion problems. The problems below should be completed in class.

(D1) *Rings of functions.* Determine which of the following sets $(R, +, \cdot)$ forms a ring under the given addition and multiplication. For each set R that is indeed a ring, determine whether R is a field.

(a) The set $R = \{\ldots, -2, 0, 2, \ldots\} \subset \mathbb{Z}$ of even integers, under the usual addition and multiplication of integers.

(b) The set $R = \{r_n x^n + \cdots + r_1 x + r_0 : r_i \in \mathbb{R}\}$ of polynomials in a variable x with real coefficients, under the usual addition and multiplication of polynomials.

(c) The subset of R from part (b) consisting of those polynomials with degree at most d.

(d) The set $R = \{f(x)/g(x) : f(x), g(x) \text{ polynomials}, g \neq 0\}$ of rational functions under the usual addition and multiplication of fractions. Here, cancelling common factors in the numerator and denominator yields the same element of R. For example,

$$\frac{x^2+5}{x} \cdot \frac{2x}{x+1} = \frac{2x(x^2+5)}{x(x+1)} = \frac{2x^2+10}{x+1}.$$

(e) The subset of R in part (d) whose denominator takes only positive values over $\mathbb{R}$.

(f) The set $R = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of continuous real-valued functions on $\mathbb{R}$, under the usual addition and multiplication of functions, e.g. $e^x \cdot \sin(x) = e^x \sin(x)$.

(g) The set $R = \mathbb{R} \cup \{\infty\}$ of real numbers together with infinity, and operations $(R, \oplus, \odot)$, where $a \oplus b = \min(a, b)$ and $a \odot b = a + b$.

(D2) *Zero divisors.* A nonzero ring element $r \in R$ is called a *zero divisor* if there exists a nonzero element $r' \in R$ with $r \cdot r' = 0$.

(a) For which n does the ring $\mathbb{Z}_n$ have zero divisors?

(b) Give an example of a ring with no zero divisors that is not a field.

(c) Which rings from problem (D1) have zero divisors?

(D3) *Cartesian products.* The Cartesian product of two rings R_1 and R_2 is the set

$$R_1 \times R_2 = \{(a, b) : a \in R_1, b \in R_2\}$$

with addition $(a, b) + (a', b') = (a + a', b + b')$ and multiplication $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$ performed componentwise.

(a) How many elements does $\mathbb{Z}_3 \times \mathbb{Z}_4$ have? What about $\mathbb{Z}_n \times \mathbb{Z}_m$?

(b) Is $\mathbb{Z}_n \times \mathbb{Z}_m$ ever a field?

(c) Under what conditions on m and n is the group $(\mathbb{Z}_n \times \mathbb{Z}_m, +)$ cyclic?

(d) The *order* of an element $a \in R$ is the smallest integer k such that adding a to itself k times yields 0 (or, written more compactly, $ka = 0$).

(i) Which elements of $\mathbb{Z}_5 \times \mathbb{Z}_3$ have highest order? What about $\mathbb{Z}_6 \times \mathbb{Z}_4$? $\mathbb{Z}_n \times \mathbb{Z}_m$?

(ii) Express the order of $(a, b) \in R_1 \times R_2$ in terms of the orders of $a \in R_1$ and $b \in R_2$.

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) Give the operation tables for all possible Abelian groups with exactly 4 elements $\{e, a, b, c\}$, where e is the identity element. Argue that there are no more. Which one is $\mathbb{Z}_4$?

(R2) Provide a short proof for each of the following statements (here, "short" means that a complete, thorough proof will likely be only a couple of lines).

(a) The group $(\mathbb{Z}_{13} \setminus \{0\}, \cdot)$ is cyclic.

(b) Fix a field F. The set $F \setminus \{0\}$ forms a group under multiplication. Is the same true if F is a ring but not a field?

(c) Fix a ring R. If $r \in R$ is a zero-divisor, then it is not invertible.

(d) If x and y are each invertible in a ring R, then xy and x^{-1} (meaning the multiplicative inverse of x) are invertible in R.

Selection problems. You are required to submit all parts of *one* selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Fix two rings R_1 and R_2.

(a) Characterize the units of $R_1 \times R_2$ in terms of the units of R_1 and R_2.

(b) Characterize the zero divisors of $R_1 \times R_2$ in terms of the zero divisors of R_1 and R_2.

(S2) The *characteristic* of a ring R is the smallest number m such that for any $r \in R$, adding r to itself m times yields 0. If no such m exists, we say R has characteristic 0.

(a) Express the characteristic of R in terms of the orders of its elements.

(b) Prove that the order of each ring element $r \in R$ divides the characteristic of R.

(c) Find the characteristic of $\mathbb{Z}_n \times \mathbb{Z}_m$ in terms of m and n.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give an example of a ring in which the set of non-unit elements is closed under both addition and multiplication.