Winter 2018, Math 148: Week 7 Problem Set
Due: Wednesday, February 28th, 2018
Block Designs

Discussion problems. The problems below should be completed in class.

(D1) *Building block designs from block designs.* Consider the following 12 subsets of $\{1, \ldots, 9\}$.

$$\begin{array}{cccc} \{1, \;\; 2, \;\; 3\} & \{1, \;\; 4, \;\; 7\} & \{1, \;\; 5, \;\; 9\} & \{1, \;\; 6, \;\; 8\} \\ \{4, \;\; 5, \;\; 6\} & \{2, \;\; 5, \;\; 8\} & \{2, \;\; 6, \;\; 7\} & \{2, \;\; 4, \;\; 9\} \\ \{7, \;\; 8, \;\; 9\} & \{3, \;\; 6, \;\; 9\} & \{3, \;\; 4, \;\; 8\} & \{3, \;\; 5, \;\; 7\} \end{array}$$

(a) For which t is the above a t-design? What are the values of v, k, r, and b in each case? Note that some of these values will be *different* for each t!

(b) Fix $s \leq t$. We saw in lecture that any t-design (v, k, r_t) is also an s-design (v, k, r_s) for some parameter r_s. Find a formula for r_1 in terms of v, k, and r_2.

(c) For $t \geq 2$, find a formula for r_{t-1} in terms of v, k, r_t, and t. Start by finding a 3-design $(v, k, r_3) = (5, 3, 1)$, and compute r_2 and r_1.

(d) Is there an 8-design (v, k, r_8) in which $r_8 = r_7 = \cdots = r_1 = 1$?

(e) Replace each set in the above example with its complement in $\{1, \ldots, 9\}$ (you should have 12 sets with 6 elements each). For which t does this form a t-design? These are called *complementary designs*. What are the values of v, k, r, and b in each case?

(f) How do these values of v, k, r, and b relate to the corresponding values in part (a)? State a conjecture about complementary 2-designs. Don't worry about proving your conjecture; you will do this on your homework!

(D2) *Designs from finite fields.* The goal of this problem is to prove the following theorem.

Theorem. *If $q = p^r$ for p prime and $r \geq 1$, there is a 2-design with $(v, k, r) = (q^2, q, 1)$.*

(a) Let's consider the case $q = 3$. Draw the 2-dimensional vector space $\mathbb{Z}_3^2$ over $\mathbb{Z}_3$.

(b) Find all lines (not necessarily containing the origin) in $\mathbb{Z}_3^2$, written as sets of points. You may find it easier to shorten points from $(2, 1)$ to simply "21" in your list.

(c) Does this collection of sets constitute a 2-design? What are v, k, r, and b?

(d) We now generalize the above construction from $\mathbb{Z}_3$ to any finite field. Suppose $\mathbb{F}_q$ is a field with q elements, let $V = \mathbb{F}_q^2$ denote a 2-dimensional vector space over $\mathbb{F}_q$, and let $B_1, \ldots, B_b \subset V$ denote the set of lines in V. For these to form a 2-design $(q^2, q, 1)$,

  (i) each block must have the same size q,
  (ii) every element must lie in the same number of blocks, and
  (iii) any pair of elements must occur together in exactly 1 block.

Restate each of the above requirements for $B_1, \ldots, B_b$ in terms of geometry.

(e) The goal of the rest of this problem is to prove the geometric statements from part (d). Any line $L \subset V$ can be written uniquely as either

$$L = \{(x, y) \in V : x = b\} \qquad \text{or} \qquad L = \{(x, y) \in V : y = ax + b\}$$

for some $a, b \in \mathbb{F}_q$. Over the finite field $\mathbb{F}_5 = \mathbb{Z}_5$, find a and b corresponding to the lines given by $\text{span}\{(1, 2)\}$, $(4, 2) + \text{span}\{(1, 2)\}$, and $(1, 4) + \text{span}\{(0, 2)\}$ in $\mathbb{F}_5^2$.

(f) Prove that any line $L \subset V$ contains exactly q points.

(g) Prove that any two lines with more than one point in common must be equal as sets. Hint: given distinct $(x_1, y_1), (x_2, y_2) \in L$, express a, b and c terms of x_1, y_1, x_2 and y_2.

(h) Find the number of lines containing each point of V (in terms of q). Hint: based on your knowledge of 2-designs, why does this follow from the other two items in part (d)?

Required problems. As the name suggests, you must submit *all* required problem with this homework set in order to receive full credit.

(R1) For each triple (v, k, r) below, construct a block design (1-design) with parameters (v, k, r), or state why no such design exists.

(a) $(v, k, r) = (7, 6, 6)$

(b) $(v, k, r) = (6, 3, 1)$

(c) $(v, k, r) = (5, 2, 1)$

(d) $(v, k, r) = (9, 6, 4)$

(R2) Suppose that there exists a 5-design with parameters $(v, k, r) = (12, 6, 1)$. Such a design is also an s-design for each $s \leq 5$; find the corresponding parameters for each s. How many blocks must this design have?

(R3) Find a 5-design with parameters $(v, k, r) = (6, 5, 1)$, or argue that no such design exists.

(R4) Is it possible there exists a 3-design with parameters $(v, k, r) = (15, 6, 2)$? What about a 4-design with parameters $(v, k, r) = (11, 5, 1)$?

(R5) Find a 2-design with parameters $(v, k, r) = (16, 4, 1)$ using the theorem in problem (D2). Hint: what field should you use, and what do the elements of that field look like?

Selection problems. You are required to submit all parts of *one* selection problem with this problem set. You may submit additional selection problems if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional selection problems.

(S1) Fix a 2-design $B_1, \ldots, B_b \subset \{1, \ldots, v\}$ with parameters (v, k, r), and let $B_i^c = \{1, \ldots, v\} \setminus B_i$ for each $i \leq b$. Prove that the blocks $B_1^c, \ldots, B_b^c$ also form a 2-design. Is the complement of a 3-design always a 3-design?

(S2) Prove that for any 2-design of the form $(v, 3, 1)$, v must have the form $6n + 1$ or $6n + 3$ for some $n \geq 0$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose $\mathbb{F}_q$ is a finite field of size q, and let $\mathbb{F}_q^3$ be a 3-dimensional vector space over $\mathbb{F}_q$. A *hyperplane* $H \subset \mathbb{F}_q^3$ is a (possibly translated) 2-dimensional subspace of $\mathbb{F}_q^3$. Any hyperplane can be expressed as

$$H = t + \text{span}\{v, w\}$$

for some $t \in \mathbb{F}_q^3$ and linearly independent $v, w \in \mathbb{F}_q^3$ (in particular, note that a hyperplane need not contain the origin).

(a) Does the collection of all hyperplanes in $\mathbb{F}_q^3$ form a 3-design? Explain using geometry.

(b) Find the number of hyperplanes in $\mathbb{F}_q^3$ (as a function of q). Prove your formula holds.